FURTHER MATHEMATICS

Paper 9231/11

Key Messages:

- Candidates need to make sure that they communicate their working in full, particularly in questions where the answer is given
- Candidates need to read the questions very carefully so that they don't miss key words such as 'hence' and 'exact' and do complete the questions fully
- Candidates should be self-critical when answers start to extend over many lines or even pages, and look back for simplifications, errors or alternative methods that they can implement
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Answer: $x = Ae^{-2t} + Bte^{-2t} - \frac{1}{2}t^2 + t + 1$

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It was good to see some very clear arguments where candidates had clearly learned how to set out their inductive proofs, and their good understanding helped them carry out all the steps correctly. Sometimes the words defining the overall structure of an induction proof were incorrect. Most candidates knew to differentiate to show the inductive step to find H_{k+1} from H_k. A few became confused with difference methods and a small number became confused between the constants and variables, but the majority differentiated correctly.

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Answer: **(i)** 4.820 **(ii)** *n* = 55

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Answer: **(i)** *r*(**M**) = 4 – 2 = 2, basis (for example) $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$ **(ii)** α = 50 , *b* = 30

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Answer: Mean Value = $\dfrac{I^3}{e-1} = \dfrac{6-2e}{e-1}$

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The first part of this question posed few problems, though a few candidates caused difficulties for themselves by substituting to form expressions for $\sin 5\theta$ in terms only of $\sin\theta$ and similarly for $\cos 5\theta$. Others omitted to show the final step of dividing by $\cos^5\theta$ which is essential when working towards a given answer. The second part of the question proved difficult to all but the strongest mathematicians, who appreciated the need to show the five roots of $\tan 5\theta = 0$ and explain how these reduced to the given roots. Candidates who remembered the appropriate trigonometric identity connecting the tangent and secant correctly picked up the final three marks without difficulty.

Answer: **(iii)** $y^2 - 12y + 16 = 0$

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This was a less popular choice than the alternative, but neatly done by most of those who chose it. Most candidates were able to find the appropriate perpendicular vectors using the cross product, and find equations for lines and planes competently. Some found the intersection of the line *AP* with the plane *OBC*, others showed more insight into the geometry.

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Answer: **(ii)** $a^2\left(\dfrac{3}{4}\pi + 2\right)$ **(iii)** $4\sqrt{2}a$

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Paper 9231/13

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| Paper 9231/21 |
| Paper 21 |

Key messages

To score full marks in the paper, candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers. Annotating a diagram in the question paper is insufficient, since this will not be seen by the Examiners.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Questions 1, 4, 7 and 10** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 1** and the directions of motion of particles, as in **Question 2**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 9 and 10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

Since the tension T in the rope and the normal reactions R_A, R_B at A, B must be found, it follows that three independent equations must be formulated. Most candidates chose to resolve forces horizontally and vertically and to take moments about a point such as A, B, C or D, though some used more than one such moment equation. A rarely-seen alternative point is the intersection of the lines of action of the two normal reactions, which has the merit of yielding T immediately. However finding the moment of T seemed to cause the most difficulty, which suggests that moments about C or D might be the most straightforward choice.

Answer: $T = \frac{1}{2}W / \sin\alpha$; $R_A = 3W/2$; $R_B = \frac{1}{2}W / \tan\alpha$.

Question 2

Almost all candidates were able to formulate and then solve two equations for the speeds v_A and v_B of the spheres A and B after the first collision by means of conservation of momentum and Newton's restitution equation. A second application of the restitution principle gives B's speed v_B' after its collision with the wall as $0{\cdot}4\,v_B$ and equating this to v_A yields the required value of e. Even though the direction of motion of B is opposite to that of A after colliding with the wall, it is necessary to equate their speeds and not their velocities, since the introduction of an inappropriate minus sign gives of course an incorrect value of e. The final part can be approached in different ways, and so candidates are advised to give some indication of what they are doing, in case an arithmetic error leads to an incorrect answer for the required distance x. Thus, for example, the time $(d - x)/v_A$ taken by A may be equated to the total time $d/v_B + x/v_B'$ taken by B. Alternatively the distance $0{\cdot}4d$ moved by A when B reaches the wall may be found from $(d/v_B)v_A$, and x must be one-half the remaining distance between A and the wall since it is given that A and B then move with the same speed before colliding. Candidates should note that applying conservation of momentum and Newton's restitution equation once again to find the speeds of A and B after their second collision is of no assistance in answering the question, and consequently earns no credit.

Answer: $\frac{1}{3}(2 - e)\,u$, $\frac{2}{3}(1 + e)\,u$; $\frac{2}{3}$; $0{\cdot}3\,d$.

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Candidates should not be confused by a superficial resemblance to a question on a previous paper, in which a particle is also attached to two elastic strings but moves vertically; here the motion is on a horizontal surface. Almost all candidates were familiar with Hooke's law, and most realised that the extensions are $0{\cdot}5a$ and $(1{\cdot}5 - k)a$ respectively when P is at the mid-point M and so were able to equate the equilibrium tensions in the two strings to find the required value of k. An application of Newton's law at a general displacement x from M gives the product of m and the acceleration of the particle in terms of the difference in tensions in the two strings, from which the standard form of the SHM equation results, with $\omega^2 = 8\,g/3a$ here. Care is needed if errors are not to be introduced in deriving this result. The period then follows as usual from $2\pi/\omega$, although it may be expressed in a variety of acceptable equivalent forms. Finally the value of a may be found from the standard SHM equation $v^2 = \omega^2(A^2 - x^2)$ with $v = 0{\cdot}7$, $A = 0{\cdot}2a$ and $x = 0{\cdot}05a$.

Answer: $1{\cdot}2$; $2\pi\surd(3a/8g)$; $0{\cdot}49$.

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The general method for finding the required 95% confidence interval was known to most candidates, though the correct unbiased estimate 103/225 ≈ 0·4578 of the population variance and the tabular *t*-value of 2·262 were not always used.

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Answer: 2/81 or 0·0247; 1/81 or 0·0123; 7.

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Most candidates appreciated that the cumulative distribution function G of *Y* is required over the interval corresponding to 1 ≤ *x* ≤ 4, which is here (*y*$^{3/2}$ – 1)/63 over 1 ≤ *y* ≤ 16, though the constant was sometimes omitted. Differentiation then gives the required non-zero expression for g(*y*), and for completeness candidates should indicate why the relevant interval is 1 ≤ *y* ≤ 16. The median value *m* of *Y* is found as usual from G(*m*) = ½, and inclusion of the constant 1/63 is of course necessary here even though its omission earlier did not affect g(*y*). Finally most candidates integrated *y*g(*y*) correctly to find the expected value of *Y*.

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Most candidates chose to use the regression line of *y* on *x* with *x* = 72 in order to estimate correctly the corresponding value of *y*, though *x* on *y* can also be used, giving an alternative estimate of 62. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form ρ = 0 and ρ > 0, though some wrongly stated them in terms of *r* which conventionally relates of course to the sample and not the population. The value 0·797 of the product moment correlation coefficient *r* was usually found correctly from the standard formula, and comparison with the tabular value 0·707 leads to a conclusion of there being evidence of non-zero correlation.

Answer: 58.

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This optional question was attempted by only a small minority of the candidates, who mostly experienced little difficulty in verifying the two given moments of inertia. These require the use of standard formulae and the parallel axes theorem where necessary to formulate and then sum the individual moments of inertia of the lamina and the rod. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. The reference in the final part of the question to the periods of small oscillations rightly suggested to most candidates attempting it that they should derive an SHM equation in each of the two cases. This is readily done by equating the product of the moment of inertia and the angular acceleration to the couple acting on the object when displaced by a small angle θ. Approximating $\sin \theta$ by θ then yields the standard form of the SHM equation and hence the period T $= 2\pi/\omega$ for each of the two cases. Finally these are verified to be equal when $m = M$. Candidates should be aware that the constant parameter ω in the SHM equation is not equal to $d\theta/dt$ even though the same symbol is sometimes used to denote the latter, and thus finding $d\theta/dt$ from conservation of energy at some fixed angular displacement and equating it to the SHM parameter ω is wholly invalid. As always, candidates should read the question carefully, noting in particular in the final part that expressions involving m and M should be found for each of the two periods, and only then setting $m = M$ rather than taking m and M to be equal from the start.

Answer: $2\pi\sqrt{\{2(8m + M)a / (3m + M)g\}}$; $2\pi\sqrt{\{(17m + M)a / 4\ mg\}}$.

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An appropriate additional assumption when comparing the Royal and Majestic plants is that their populations share a common variance. Although an explicit statement of this assumption was often omitted, the majority of the many candidates attempting this optional question did indeed attempt to find a pooled estimate 0·431 of this common variance, and hence a calculated t-value of 2·13. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. Since it is a two-tailed test, comparison with the tabulated value of 2·086 leads to acceptance of the alternative hypothesis, namely that the population mean masses are not the same. The second part proved more challenging, possibly because many candidates were unsure how to estimate an appropriate variance. Since nothing is known about the Crown plants apart from their mean, the estimate of the population variance obtained from the 10 Royal plants should be used, giving a calculated t-value of 2. Comparison with the corresponding tabular value of 1·833 again leads to acceptance of the alternative hypothesis, namely here that the population mean mass is greater than 3·8, so Farmer *A*'s claim is justified.

FURTHER MATHEMATICS

| Paper 9231/22 |
| Paper 22 |

Key messages

To score full marks in the paper, candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

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Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

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Almost all candidates were able to formulate and then solve two equations for the speeds v_A and v_B of the spheres A and B after the first collision by means of conservation of momentum and Newton's restitution equation. A second application of the restitution principle gives B's speed v_B' after its collision with the wall as $0.4\,v_B$ and equating this to v_A yields the required value of e. Even though the direction of motion of B is opposite to that of A after colliding with the wall, it is necessary to equate their speeds and not their velocities, since the introduction of an inappropriate minus sign gives of course an incorrect value of e. The final part can be approached in different ways, and so candidates are advised to give some indication of what they are doing, in case an arithmetic error leads to an incorrect answer for the required distance x. Thus, for example, the time $(d - x)/v_A$ taken by A may be equated to the total time $d/v_B + x/v_B'$ taken by B. Alternatively the distance $0.4d$ moved by A when B reaches the wall may be found from $(d/v_B)v_A$, and x must be one-half the remaining distance between A and the wall since it is given that A and B then move with the same speed before colliding. Candidates should note that applying conservation of momentum and Newton's restitution equation once again to find the speeds of A and B after their second collision is of no assistance in answering the question, and consequently earns no credit.

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FURTHER MATHEMATICS

Paper 9231/23
Paper 23

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Answer: 58.

Question 10 (Mechanics)

This optional question was attempted by only a small minority of the candidates, who mostly experienced little difficulty in verifying the two given moments of inertia. These require the use of standard formulae and the parallel axes theorem where necessary to formulate and then sum the individual moments of inertia of the lamina and the rod. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. The reference in the final part of the question to the periods of small oscillations rightly suggested to most candidates attempting it that they should derive an SHM equation in each of the two cases. This is readily done by equating the product of the moment of inertia and the angular acceleration to the couple acting on the object when displaced by a small angle θ. Approximating $\sin\theta$ by θ then yields the standard form of the SHM equation and hence the period $T = 2\pi/\omega$ for each of the two cases. Finally these are verified to be equal when $m = M$. Candidates should be aware that the constant parameter ω in the SHM equation is not equal to $d\theta/dt$ even though the same symbol is sometimes used to denote the latter, and thus finding $d\theta/dt$ from conservation of energy at some fixed angular displacement and equating it to the SHM parameter ω is wholly invalid. As always, candidates should read the question carefully, noting in particular in the final part that expressions involving m and M should be found for each of the two periods, and only then setting $m = M$ rather than taking m and M to be equal from the start.

Answer: $2\pi\sqrt{\{2(8m + M)a / (3m + M)g\}}$; $2\pi\sqrt{\{(17m + M)a / 4\,mg\}}$.

Question 10 (Statistics)

An appropriate additional assumption when comparing the Royal and Majestic plants is that their populations share a common variance. Although an explicit statement of this assumption was often omitted, the majority of the many candidates attempting this optional question did indeed attempt to find a pooled estimate 0·431 of this common variance, and hence a calculated t-value of 2·13. As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. Since it is a two-tailed test, comparison with the tabulated value of 2·086 leads to acceptance of the alternative hypothesis, namely that the population mean masses are not the same. The second part proved more challenging, possibly because many candidates were unsure how to estimate an appropriate variance. Since nothing is known about the Crown plants apart from their mean, the estimate of the population variance obtained from the 10 Royal plants should be used, giving a calculated t-value of 2. Comparison with the corresponding tabular value of 1·833 again leads to acceptance of the alternative hypothesis, namely here that the population mean mass is greater than 3·8, so Farmer A's claim is justified.